

## ON BACH-FLAT GRADIENT SHRINKING RICCI SOLITONS

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ABSTRACT. In this paper, we classify  $n$ -dimensional ( $n \geq 4$ ) complete Bach-flat gradient shrinking Ricci solitons. More precisely, we prove that any 4-dimensional Bach-flat gradient shrinking Ricci soliton is either Einstein, or locally conformally flat hence a finite quotient of the Gaussian shrinking soliton  $\mathbb{R}^4$  or the round cylinder  $\mathbb{S}^3 \times \mathbb{R}$ . More generally, for  $n \geq 5$ , a Bach-flat gradient shrinking Ricci soliton is either Einstein, or a finite quotient of the Gaussian shrinking soliton  $\mathbb{R}^n$  or the product  $N^{n-1} \times \mathbb{R}$ , where  $N^{n-1}$  is Einstein.

## 1. THE RESULTS

A complete Riemannian manifold  $(M^n, g_{ij})$  is called a *gradient Ricci soliton* if there exists a smooth function  $f$  on  $M^n$  such that the Ricci tensor  $R_{ij}$  of the metric  $g_{ij}$  satisfies the equation

$$R_{ij} + \nabla_i \nabla_j f = \rho g_{ij}$$

for some constant  $\rho$ . For  $\rho = 0$  the Ricci soliton is *steady*, for  $\rho > 0$  it is *shrinking* and for  $\rho < 0$  *expanding*. The function  $f$  is called a *potential function* of the gradient Ricci soliton. Clearly, when  $f$  is a constant the gradient Ricci soliton is simply an Einstein manifold. Thus Ricci solitons are natural extensions of Einstein metrics. Gradient Ricci solitons play an important role in Hamilton's Ricci flow as they correspond to self-similar solutions, and often arise as singularity models. Therefore it is important to classify gradient Ricci solitons or understand their geometry.

In this paper we shall focus our attention on gradient shrinking Ricci solitons, which are possible Type I singularity models in the Ricci flow. We normalize the constant  $\rho = 1/2$  so that the shrinking soliton equation is given by

$$R_{ij} + \nabla_i \nabla_j f = \frac{1}{2} g_{ij}. \quad (1.1)$$

In recent years, inspired by Perelman's work [22, 23], much efforts have been devoted to study the geometry and classifications of gradient shrinking Ricci solitons. We refer the reader to the survey papers [4, 5] by the first author and the references therein for recent progress on the subject. In particular, it is known (cf. [23, 21, 7]) that any complete 3-dimensional gradient shrinking Ricci soliton is a finite quotient of either the round sphere  $\mathbb{S}^3$ , or the Gaussian shrinking soliton  $\mathbb{R}^3$ , or the round cylinder  $\mathbb{S}^2 \times \mathbb{R}$ . For higher dimensions, it has been proven that complete locally conformally flat gradient shrinking Ricci solitons are finite quotients of either the round sphere  $\mathbb{S}^n$ , or the Gaussian shrinking soliton  $\mathbb{R}^n$ , or the round cylinder  $\mathbb{S}^{n-1} \times \mathbb{R}$  (first due to Z. H. Zhang [26] based on the work of Ni-Wallach [21], see also

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the works of Eminenti-La Nave-Mantegazza [15], Petersen-Wylie [24], X. Cao, B. Wang and Z. Zhang [10], and Munteanu-Sesum [20]). Moreover, it follows from the works of Fernández-López and García-Río [16], and Munteanu-Sesum [20] that  $n$ -dimensional complete gradient shrinking solitons with harmonic Weyl tensor are rigid in the sense that they are finite quotients of the product of an Einstein manifold  $N^k$  with the Gaussian shrinking soliton  $\mathbb{R}^{n-k}$ .

Our aim in this paper is to investigate an interesting class of complete gradient shrinking Ricci solitons: those with vanishing Bach tensor. This well-known tensor was introduced by R. Bach [1] in early 1920s' to study conformal relativity. On any  $n$ -dimensional manifold  $(M^n, g_{ij})$  ( $n \geq 4$ ), the Bach tensor is defined by

$$B_{ij} = \frac{1}{n-3} \nabla^k \nabla^l W_{ikjl} + \frac{1}{n-2} R_{kl} W_i{}^k{}_j{}^l.$$

Here  $W_{ikjl}$  is the Weyl tensor. It is easy to see that if  $(M^n, g_{ij})$  is either locally conformally flat (i.e.,  $W_{ikjl} = 0$ ) or Einstein, then  $(M^n, g_{ij})$  is *Bach-flat*:  $B_{ij} = 0$ .

The case when  $n = 4$  is the most interesting, as it is well-known (cf. [2] or [14]) that on any compact 4-manifold  $(M^4, g_{ij})$ , Bach-flat metrics are precisely the critical points of the following *conformally invariant* functional on the space of metrics,

$$\mathcal{W}(g) = \int_M |W_g|^2 dV_g,$$

where  $W_g$  denotes the Weyl tensor of  $g$ . Moreover, if  $(M^4, g_{ij})$  is either *half conformally flat* (i.e., self-dual or anti-self-dual) or *locally conformal to an Einstein manifold*, then its Bach tensor vanishes. In this paper, we shall see the (stronger) converse holds for gradient shrinking solitons: *Bach-flat 4-dimensional gradient shrinking solitons are either Einstein or locally conformally flat*.

Our main results are the following classification theorems for Bach-flat gradient shrinking Ricci solitons:

**Theorem 1.1.** *Let  $(M^4, g_{ij}, f)$  be a complete Bach-flat gradient shrinking Ricci soliton. Then,  $(M^4, g_{ij}, f)$  is either*

- (i) *Einstein, or*
- (ii) *locally conformally flat, hence a finite quotient of either the Gaussian shrinking soliton  $\mathbb{R}^4$  or the round cylinder  $\mathbb{S}^3 \times \mathbb{R}$ .*

More generally, for  $n \geq 5$ , we have:

**Theorem 1.2.** *Let  $(M^n, g_{ij}, f)$  ( $n \geq 5$ ) be a complete Bach-flat gradient shrinking Ricci soliton. Then,  $(M^n, g_{ij}, f)$  is either*

- (i) *Einstein, or*
- (ii) *a finite quotient of the Gaussian shrinking soliton  $\mathbb{R}^n$ , or*
- (iii) *a finite quotient of  $N^{n-1} \times \mathbb{R}$ , where  $N^{n-1}$  is an Einstein manifold of positive scalar curvature.*

The basic idea in proving Theorem 1.1 and Theorem 1.2 is to explore hidden relations between the Bach tensor  $B_{ij}$  and the Cotton tensor  $C_{ijk}$  on a gradient

shrinking Ricci soliton. It turns out that the key link between these two classical tensors is provided by a third tensor, the covariant 3-tensor  $D_{ijk}$  defined by

$$D_{ijk} = \frac{1}{n-2}(A_{jk}\nabla_i f - A_{ik}\nabla_j f) + \frac{1}{(n-1)(n-2)}(g_{jk}E_{il} - g_{ik}E_{jl})\nabla_l f, \quad (1.2)$$

where  $A_{ij}$  is the Schouten tensor and  $E_{ij}$  is the Einstein tensor (see Section 3).

This tensor  $D_{ijk}$  (and its equivalent version in Section 3) was introduced by the authors in [8] to study the classification of locally conformally flat gradient steady solitons. On one hand, for any gradient Ricci soliton, it turns out that the Bach tensor  $B_{ij}$  can be expressed in terms of  $D_{ijk}$  and the Cotton tensor  $C_{ijk}$ :

$$B_{ij} = -\frac{1}{n-2}(\nabla_k D_{ikj} + \frac{n-3}{n-2}C_{jli}\nabla_l f). \quad (1.3)$$

On the other hand, as shown in [8],  $D_{ijk}$  is closely related to the Cotton tensor and the Weyl tensor by

$$D_{ijk} = C_{ijk} + W_{ijkl}\nabla_l f. \quad (1.4)$$

By using (1.3), we are able to show that the vanishing of the Bach tensor  $B_{ij}$  implies the vanishing of  $D_{ijk}$  for gradient shrinking solitons (see Lemma 4.1). On the other hand, the norm of  $D_{ijk}$  is linked to the geometry of level surfaces of the potential function  $f$  by the following key identity (see Proposition 3.1): at any point  $p \in M^n$  where  $\nabla f(p) \neq 0$ ,

$$|D_{ijk}|^2 = \frac{2|\nabla f|^4}{(n-2)^2}|h_{ab} - \frac{H}{n-1}g_{ab}|^2 + \frac{1}{2(n-1)(n-2)}|\nabla_a R|^2, \quad (1.5)$$

where  $h_{ab}$  and  $H$  are the second fundamental form and the mean curvature for the level surface  $\Sigma = \{f = f(p)\}$ , and  $g_{ab}$  is the induced metric on the level surface  $\Sigma$ . Thus, the vanishing of  $D_{ijk}$  and (1.5) tell us that the geometry of the shrinking Ricci soliton and the level surfaces of the potential function are very special (see Proposition 3.2), and consequently we deduce that  $D_{ijk} = 0$  implies the Cotton tensor  $C_{ijk} = 0$  at all points where  $|\nabla f| \neq 0$  (see Lemma 4.2 and Theorem 5.1). Furthermore, when  $n = 4$ , by using (1.4) we can actually show that the Weyl tensor  $W_{ijkl}$  must vanish at all points where  $|\nabla f| \neq 0$  (see Lemma 4.3). Then the main theorems follow immediately from the known classification theorem for locally conformally flat gradient shrinking Ricci solitons and the rigid theorem for gradient shrinking Ricci solitons with harmonic Weyl tensor respectively.

*Remark 1.1.* Very recently, by cleverly using the tensor  $D_{ijk}$ , X. Chen and Y. Wang [13] have shown that 4-dimensional half-conformally flat gradient shrinking Ricci solitons are either Einstein, or locally conformally flat. Since half-conformal flat implies Bach-flat in dimension 4, our Theorem 1.1 is clearly an improvement.

Note that by a theorem of Hitchin (cf. [2], Theorem 13.30), a compact 4-dimensional half-conformally flat Einstein manifold (of positive scalar curvature) is  $\mathbb{S}^4$  or  $\mathbb{C}P^2$ . Combining Hitchin's theorem and Theorem 1.1, we arrive at the following classification of 4-dimensional compact half-conformally flat gradient shrinking Ricci solitons which was first obtained by X. Chen and Y. Wang [13]:

**Corollary 1.1.** *If  $(M^4, g_{ij}, f)$  is a compact half-conformally flat gradient shrinking Ricci soliton, then  $(M^4, g_{ij})$  is isometric to the standard  $\mathbb{S}^4$  or  $\mathbb{C}P^2$ .*

Finally, in Section 5, we observe that for all gradient (shrinking, or steady, or expanding) Ricci solitons, the vanishing of  $D_{ijk}$  implies the vanishing of the Cotton tensor  $C_{ijk}$  at all points where  $|\nabla f| \neq 0$  (see Theorem 5.1). This yields the classification of  $n$ -dimensional ( $n \geq 4$ ) gradient shrinking Ricci solitons, as well as 4-dimensional gradient steady Ricci solitons, with vanishing  $D_{ijk}$ .

**Theorem 1.3.** *Let  $(M^n, g_{ij}, f)$  ( $n \geq 4$ ) be a complete gradient shrinking Ricci soliton with  $D_{ijk} = 0$ , then*

(i)  *$(M^4, g_{ij}, f)$  is either Einstein, or a finite quotient of  $\mathbb{R}^4$  or  $\mathbb{S}^3 \times \mathbb{R}$ ;*

(ii) *for  $n \geq 5$ ,  $(M^n, g_{ij}, f)$  is either Einstein, or a finite quotient of the Gaussian shrinking soliton  $\mathbb{R}^n$ , or a finite quotient of  $N^{n-1} \times \mathbb{R}$ , where  $N^{n-1}$  is Einstein.*

**Theorem 1.4.** *Let  $(M^4, g_{ij}, f)$  be a complete gradient steady Ricci soliton with  $D_{ijk} = 0$ , then  $(M^4, g_{ij}, f)$  is either Ricci flat or isometric to the Bryant soliton.*

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## 2. PRELIMINARIES

In this section, we fix our notations and recall some basic facts and known results about gradient Ricci solitons that we shall need in the proof of Theorem 1.1 and Theorem 1.2.

First of all, we recall that on any  $n$ -dimensional Riemannian manifold  $(M^n, g_{ij})$  ( $n \geq 3$ ), the Weyl curvature tensor is given by

$$\begin{aligned} W_{ijkl} = & R_{ijkl} - \frac{1}{n-2}(g_{ik}R_{jl} - g_{il}R_{jk} - g_{jk}R_{il} + g_{jl}R_{ik}) \\ & + \frac{R}{(n-1)(n-2)}(g_{ik}g_{jl} - g_{il}g_{jk}), \end{aligned}$$

and the Cotton tensor by

$$C_{ijk} = \nabla_i R_{jk} - \nabla_j R_{ik} - \frac{1}{2(n-1)}(g_{jk}\nabla_i R - g_{ik}\nabla_j R).$$

*Remark 2.1.* In terms of the Schouten tensor

$$A_{ij} = R_{ij} - \frac{R}{2(n-1)}g_{ij}, \tag{2.1}$$

$$\begin{aligned} W_{ijkl} = & R_{ijkl} - \frac{1}{n-2}(g_{ik}A_{jl} - g_{il}A_{jk} - g_{jk}A_{il} + g_{jl}A_{ik}), \\ C_{ijk} = & \nabla_i A_{jk} - \nabla_j A_{ik}. \end{aligned}$$

It is well known that, for  $n = 3$ ,  $W_{ijkl}$  vanishes identically, while  $C_{ijk} = 0$  if and only if  $(M^3, g_{ij})$  is locally conformally flat; for  $n \geq 4$ ,  $W_{ijkl} = 0$  if and only if  $(M^n, g_{ij})$  is locally conformally flat. Moreover, for  $n \geq 4$ , the Cotton tensor  $C_{ijk}$  is, up to a constant factor, the divergence of the Weyl tensor:

$$C_{ijk} = -\frac{n-2}{n-3}\nabla_l W_{ijkl}, \quad (2.2)$$

hence the vanishing of the Cotton tensor  $C_{ijk} = 0$  (in dimension  $n \geq 4$ ) is also referred as being harmonic Weyl.

Moreover, for  $n \geq 4$ , the Bach tensor is defined by

$$B_{ij} = \frac{1}{n-3}\nabla^k \nabla^l W_{ikjl} + \frac{1}{n-2}R_{kl}W_i{}^k{}_j{}^l.$$

By (2.2), we have

$$B_{ij} = \frac{1}{n-2}(\nabla_k C_{kij} + R_{kl}W_i{}^k{}_j{}^l). \quad (2.3)$$

Note that  $C_{ijk}$  is skew-symmetric in the first two indices and trace-free in any two indices:

$$C_{ijk} = -C_{jik} \quad \text{and} \quad g^{ij}C_{ijk} = g^{ik}C_{ijk} = 0. \quad (2.4)$$

Next we recall some basic facts about complete gradient shrinking Ricci solitons satisfying Eq. (1.1).

**Lemma 2.1. (Hamilton [18])** *Let  $(M^n, g_{ij}, f)$  be a complete gradient shrinking Ricci soliton satisfying Eq. (1.1). Then we have*

$$\nabla_i R = 2R_{ij}\nabla_j f, \quad (2.5)$$

and

$$R + |\nabla f|^2 - f = C_0$$

for some constant  $C_0$ . Here  $R$  denotes the scalar curvature.

Note that if we normalize  $f$  by adding the constant  $C_0$  to it, then we have

$$R + |\nabla f|^2 = f. \quad (2.6)$$

**Lemma 2.2.** *Let  $(M^n, g_{ij}, f)$  be a complete gradient steady soliton. Then it has nonnegative scalar curvature  $R \geq 0$ .*

Lemma 2.2 is a special case of a more general result of B.-L. Chen [12] which states that  $R \geq 0$  for any ancient solution to the Ricci flow. For an alternative proof of Lemma 2.2, see, e.g., the more recent work of Pigola-Rimoldi-Setti [25].

**Lemma 2.3. (Cao-Zhou [9])** *Let  $(M^n, g_{ij}, f)$  be a complete noncompact gradient shrinking Ricci soliton satisfying (1.1) and the normalization (2.6). Then,*

(i) *the potential function  $f$  satisfies the estimates*

$$\frac{1}{4}(r(x) - c_1)^2 \leq f(x) \leq \frac{1}{4}(r(x) + c_2)^2,$$

where  $r(x) = d(x_0, x)$  is the distance function from some fixed point  $x_0 \in M$ ,  $c_1$  and  $c_2$  are positive constants depending only on  $n$  and the geometry of  $g_{ij}$  on the unit ball  $B(x_0, 1)$ ;

(ii) *there exists some constant  $C > 0$  such that*

$$\text{Vol}(B(x_0, s)) \leq Cs^n$$

for  $s > 0$  sufficiently large.

3. THE COVARIANT 3-TENSOR  $D_{ijk}$ 

In this section, we review the covariant 3-tensor  $D_{ijk}$  introduced in our previous work [8] and its important properties.

For any gradient Ricci soliton satisfying the defining equation

$$R_{ij} + \nabla_i \nabla_j f = \rho g_{ij}, \quad (3.1)$$

the covariant 3-tensor  $D_{ijk}$  is defined as:

$$\begin{aligned} D_{ijk} = & \frac{1}{n-2} (R_{jk} \nabla_i f - R_{ik} \nabla_j f) + \frac{1}{2(n-1)(n-2)} (g_{jk} \nabla_i R - g_{ik} \nabla_j R) \\ & - \frac{R}{(n-1)(n-2)} (g_{jk} \nabla_i f - g_{ik} \nabla_j f). \end{aligned}$$

Note that, by using (2.5),  $D_{ijk}$  can also be expressed as

$$D_{ijk} = \frac{1}{n-2} (A_{jk} \nabla_i f - A_{ik} \nabla_j f) + \frac{1}{(n-1)(n-2)} (g_{jk} E_{il} - g_{ik} E_{jl}) \nabla_l f, \quad (3.2)$$

where  $A_{ij}$  is the Schouten tensor in (2.1) and  $E_{ij} = R_{ij} - \frac{R}{2} g_{ij}$  is the Einstein tensor.

This 3-tensor  $D_{ijk}$  is closely tied to the Cotton tensor and played a significant role in our previous work [8] on classifying locally conformally flat gradient steady solitons, as well as in the subsequent works of S. Brendle [3] and X. Chen and Y. Wang [13].

**Lemma 3.1.** *Let  $(M^n, g_{ij}, f)$  ( $n \geq 3$ ) be a complete gradient soliton satisfying (3.1). Then  $D_{ijk}$  is related to the Cotton tensor  $C_{ijk}$  and the Weyl tensor  $W_{ijkl}$  by*

$$D_{ijk} = C_{ijk} + W_{ijkl} \nabla_l f.$$

*Proof.* From the soliton equation (3.1), we have

$$\nabla_i R_{jk} - \nabla_j R_{ik} = -\nabla_i \nabla_j \nabla_k f + \nabla_j \nabla_i \nabla_k f = -R_{ijkl} \nabla_l f.$$

Hence, using (2.5), we obtain

$$\begin{aligned} C_{ijk} = & \nabla_i R_{jk} - \nabla_j R_{ik} - \frac{1}{2(n-1)} (g_{jk} \nabla_i R - g_{ik} \nabla_j R) \\ = & -R_{ijkl} \nabla_l f - \frac{1}{(n-1)} (g_{jk} R_{il} - g_{ik} R_{jl}) \nabla_l f \\ = & -W_{ijkl} \nabla_l f - \frac{1}{n-2} (R_{ik} \nabla_j f - R_{jk} \nabla_i f) \\ & + \frac{1}{2(n-1)(n-2)} (g_{jk} \nabla_i R - g_{ik} \nabla_j R) + \frac{R}{(n-1)(n-2)} (g_{ik} \nabla_j f - g_{jk} \nabla_i f) \\ = & -W_{ijkl} \nabla_l f + D_{ijk}. \end{aligned}$$

□

*Remark 3.1.* By Lemma 3.1,  $D_{ijk}$  is equal to the Cotton tensor  $C_{ijk}$  in dimension  $n = 3$ . In addition, it is easy to see that

$$D_{ijk} \nabla_k f = C_{ijk} \nabla_k f$$

Also,  $D_{ijk}$  vanishes if  $(M^n, g_{ij}, f)$  ( $n \geq 3$ ) is either Einstein or locally conformally flat. Moreover, like the Cotton tensor  $C_{ijk}$ ,  $D_{ijk}$  is skew-symmetric in the first two indices and trace-free in any two indices:

$$D_{ijk} = -D_{jik} \quad \text{and} \quad g^{ij} D_{ijk} = g^{ik} D_{ijk} = 0. \quad (3.3)$$

What is so special about  $D_{ijk}$  is the following key identity, which links the norm of  $D_{ijk}$  to the geometry of the level surfaces of the potential function  $f$ . We refer readers to [8] (cf. Lemma 4.4 in [8]) for its proof.

**Proposition 3.1. (Cao-Chen [8])** *Let  $(M^n, g_{ij}, f)$  ( $n \geq 3$ ) be an  $n$ -dimensional gradient Ricci soliton satisfying (3.1). Then, at any point  $p \in M^n$  where  $\nabla f(p) \neq 0$ , we have*

$$|D_{ijk}|^2 = \frac{2|\nabla f|^4}{(n-2)^2} |h_{ab} - \frac{H}{n-1} g_{ab}|^2 + \frac{1}{2(n-1)(n-2)} |\nabla_a R|^2,$$

where  $h_{ab}$  and  $H$  are the second fundamental form and the mean curvature of the level surface  $\Sigma = \{f = f(p)\}$ , and  $g_{ab}$  is the induced metric on  $\Sigma$ .

Finally, thanks to Proposition 3.1, the vanishing of  $D_{ijk}$  implies many nice properties about the geometry of the Ricci soliton  $(M^n, g_{ij}, f)$  and the level surfaces of the potential function  $f$ .

**Proposition 3.2.** *Let  $(M^n, g_{ij}, f)$  ( $n \geq 3$ ) be any complete gradient Ricci soliton with  $D_{ijk} = 0$ , and let  $c$  be a regular value of  $f$  and  $\Sigma_c = \{f = c\}$  be the level surface of  $f$ . Set  $e_1 = \nabla f/|\nabla f|$  and pick any orthonormal frame  $e_2, \dots, e_n$  tangent to the level surface  $\Sigma_c$ . Then*

- (a)  $|\nabla f|^2$  and the scalar curvature  $R$  of  $(M^n, g_{ij}, f)$  are constant on  $\Sigma_c$ ;
- (b)  $R_{1a} = 0$  for any  $a \geq 2$  and  $e_1 = \nabla f/|\nabla f|$  is an eigenvector of  $Rc$ ;
- (c) the second fundamental form  $h_{ab}$  of  $\Sigma_c$  is of the form  $h_{ab} = \frac{H}{n-1} g_{ab}$ ;
- (d) the mean curvature  $H$  is constant on  $\Sigma_c$ ;
- (e) on  $\Sigma_c$ , the Ricci tensor of  $(M^n, g_{ij}, f)$  either has a unique eigenvalue  $\lambda$ , or has two distinct eigenvalues  $\lambda$  and  $\mu$  of multiplicity 1 and  $n-1$  respectively. In either case,  $e_1 = \nabla f/|\nabla f|$  is an eigenvector of  $\lambda$

*Proof.* Clearly (a) and (c) follow immediately from  $D_{ijk} = 0$ , Proposition 3.1, and (2.6);

(b) follows from (a) and (2.5):  $R_{1a} = \frac{1}{2|\nabla f|} \nabla_a R = 0$ ;

For (d), we consider the Codazzi equation

$$R_{1cab} = \nabla_a^{\Sigma_c} h_{bc} - \nabla_b^{\Sigma_c} h_{ac}, \quad a, b, c = 2, \dots, n. \quad (3.4)$$

Tracing over  $b$  and  $c$  in (3.4), we obtain

$$R_{1a} = \nabla_a^{\Sigma_c} H - \nabla_b^{\Sigma_c} h_{ab} = (1 - \frac{1}{n-1}) \nabla_a H.$$

Then (d) follows since  $R_{1a} = 0$ ;

Finally, the second fundamental form is given by

$$h_{ab} = \langle \nabla_a \frac{\nabla f}{|\nabla f|}, e_b \rangle = \frac{\nabla_a \nabla_b f}{|\nabla f|} = \frac{\rho g_{ab} - R_{ab}}{|\nabla f|}.$$

Combining this with (c), we see that

$$R_{ab} = \rho g_{ab} - |\nabla f| h_{ab} = (\rho - \frac{H}{n-1} |\nabla f|) g_{ab}.$$

But both  $H$  and  $|\nabla f|$  are constant on  $\Sigma_c$ , so the Ricci tensor restricted to the tangent space of  $\Sigma_c$  has only one eigenvalue  $\mu$ :

$$\mu = R_{aa} = \rho - H|\nabla f|/(n-1), \quad a = 2, \dots, n, \quad (3.5)$$

which is constant along  $\Sigma_c$ . On the other hand,

$$\lambda = R_{11} = R - \sum_{a=2}^n R_{aa} = R - (n-1)\rho + H|\nabla f|, \quad (3.6)$$

again a constant along  $\Sigma_c$ . This proves (e).  $\square$

*Remark 3.2.* In any neighborhood  $U$  of the level surface  $\Sigma_c$  where  $|\nabla f|^2 \neq 0$ , we can always express the metric  $g_{ij}$  as

$$ds^2 = \frac{1}{|\nabla f|^2(f, \theta)} (df)^2 + g_{ab}(f, \theta) d\theta^a d\theta^b. \quad (3.7)$$

Here  $\theta = (\theta^2, \dots, \theta^n)$  denotes any local coordinates on  $\Sigma_c$ . It follows from Proposition 3.2 that, when  $D_{ijk} = 0$ , the metric  $g_{ij}$  is in fact a warped product metric on  $U$  of the form:

$$ds^2 = dr^2 + \varphi^2(r) \bar{g}_{\Sigma_c}, \quad (3.8)$$

where  $\bar{g}_{\Sigma_c}$  denotes the induced metric on  $\Sigma_c$ . Furthermore,  $(\Sigma_c, \bar{g}_{\Sigma_c})$  is necessarily Einstein. The details can be found in [6].

#### 4. THE PROOF OF THEOREM 1.1 AND THEOREM 1.2

Throughout this section, we assume that  $(M^n, g_{ij}, f)$  ( $n \geq 4$ ) is a complete gradient shrinking soliton satisfying (1.1).

First of all, we relate the Bach tensor  $B_{ij}$  to the Cotton tensor  $C_{ijk}$  and the tensor  $D_{ijk}$ , and then show that the Bach-flatness implies  $D_{ijk} = 0$ .

**Lemma 4.1.** *Let  $(M^n, g_{ij}, f)$  be a complete gradient shrinking soliton. If  $B_{ij} = 0$ , then  $D_{ijk} = 0$ .*

*Proof.* By direct computations, and using (2.2), (2.3) and Lemma 3.1, we have

$$\begin{aligned} B_{ij} &= -\frac{1}{n-2} \nabla_k C_{ikj} + \frac{1}{n-2} R_{kl} W_{ikjl} \\ &= -\frac{1}{n-2} \nabla_k (D_{ikj} - W_{ikjl} \nabla_l f) + \frac{1}{n-2} R_{kl} W_{ikjl} \\ &= -\frac{1}{n-2} (\nabla_k D_{ikj} - \nabla_k W_{jlik} \nabla_l f) + \frac{1}{n-2} (R_{kl} + \nabla_k \nabla_l f) W_{ijkl}. \end{aligned}$$

Hence,

$$B_{ij} = -\frac{1}{n-2} (\nabla_k D_{ikj} + \frac{n-3}{n-2} C_{jli} \nabla_l f). \quad (4.1)$$

Next, we use (4.1) to show that Bach flatness implies the vanishing of the tensor  $D_{ijk}$ . By Lemma 2.3, for each  $r > 0$  sufficiently large,  $\Omega_r = \{x \in M | f(x) \leq r\}$



is compact. Now by the definition of  $D_{ijk}$ , the identity (4.1), as well as properties (2.4) and (3.3), we have

$$\begin{aligned}
\int_{\Omega_r} B_{ij} \nabla_i f \nabla_j f dV &= -\frac{1}{(n-2)} \int_{\Omega_r} \nabla_k D_{ikj} \nabla_i f \nabla_j f dV \\
&= \frac{1}{(n-2)} \left( \int_{\Omega_r} D_{ikj} \nabla_i f \nabla_k \nabla_j f dV - \int_{\Omega_r} \nabla_k (D_{ikj} \nabla_i f \nabla_j f) dV \right) \\
&= -\frac{1}{(n-2)} \left( \int_{\Omega_r} D_{ikj} \nabla_i f R_{jk} dV + \int_{\partial\Omega_r} D_{ikj} \nabla_i f \nabla_j f \nu_k dS \right) \\
&= -\frac{1}{2(n-2)} \int_{\Omega_r} D_{ikj} (\nabla_i f R_{jk} - \nabla_k f R_{ij}) dV \\
&= -\frac{1}{2} \int_{\Omega_r} |D_{ikj}|^2 dV.
\end{aligned}$$

Here we have used the fact, in view of (3.3), that

$$\int_{\partial\Omega_r} D_{ikj} \nabla_i f \nabla_j f \nu_k dS = \int_{\partial\Omega_r} D_{ikj} \nabla_i f \nabla_j f \nabla_k f \frac{1}{|\nabla f|} dS = 0.$$

By taking  $r \rightarrow \infty$ , we immediately obtain

$$\int_M B_{ij} \nabla_i f \nabla_j f dV = -\frac{1}{2} \int_M |D_{ikj}|^2 dV.$$

This completes the proof of Lemma 4.1.  $\square$

**Lemma 4.2.** *Let  $(M^n, g_{ij}, f)$  ( $n \geq 4$ ) be a complete gradient shrinking Ricci soliton with vanishing  $D_{ijk}$ , then the Cotton tensor  $C_{ijk} = 0$  at all points where  $\nabla f \neq 0$ .*

*Proof.* First of all,  $D_{ijk} = 0$  and Lemma 3.1 imply

$$C_{ijk} = -W_{ijkl} \nabla_l f, \quad (4.2)$$

hence

$$C_{ijk} \nabla_k f = -W_{ijkl} \nabla_k f \nabla_l f = 0. \quad (4.3)$$

Next, for any point  $p \in M$  with  $\nabla f(p) \neq 0$ , we choose a local coordinates system  $(\theta^2, \dots, \theta^n)$  on the level surface  $\Sigma = \{f = f(p)\}$ . In any neighborhood  $U$  of the level surface  $\Sigma$  where  $|\nabla f|^2 \neq 0$ , we use the local coordinates system

$$(x^1, x^2, \dots, x^n) = (f, \theta^2, \dots, \theta^n)$$

adapted to level surfaces. In the following, we use  $a, b, c$  to represent indices on the level sets which ranges from 2 to  $n$ , while  $i, j, k$  from 1 to  $n$ . Under the above chosen local coordinates system, the metric  $g$  can be expressed as

$$ds^2 = \frac{1}{|\nabla f|^2} df^2 + g_{ab}(f, \theta) d\theta^a d\theta^b.$$

Next, we denote by  $\nu = -\frac{\nabla f}{|\nabla f|}$ . It is then easy to see that

$$\nu = -|\nabla f| \partial_f, \quad \text{or} \quad \partial_f = \frac{1}{|\nabla f|^2} \nabla f.$$

Also  $\partial_1$  and  $\partial_f$  shall be interchangeable below. And we have

$$\nabla_1 f = 1, \quad \text{and} \quad \nabla_a f = 0 \text{ for } a \geq 2. \quad (4.4)$$

Then, in this coordinate, (4.3) implies that

$$C_{ij1} = 0.$$

**Claim 1:**  $D_{ijk} = 0$  implies  $C_{abc} = 0$  for  $a \geq 2, b \geq 2$ , and  $c \geq 2$ .

To show  $C_{abc} = 0$ , we make use of Proposition 3.2 as follows: from the Codazzi equation (3.4) and  $h_{ab} = Hg_{ab}/(n-1)$ , we get

$$R_{1cab} = \nabla_a^\Sigma h_{bc} - \nabla_b^\Sigma h_{ac} = \frac{1}{n-1}(g_{bc}\partial_a(H) - g_{ac}\partial_b(H)). \quad (4.5)$$

But we also know that the mean curvature  $H$  is constant on the level surface  $\Sigma$  of  $f$ , so

$$R_{1abc} = 0.$$

Moreover, since  $R_{1a} = 0$ , we easily obtain

$$W_{1abc} = R_{1abc} = 0.$$

By (4.2), we have

$$C_{abc} = -W_{abci}\nabla_j f g^{ij} = W_{1cab}\nabla_1 f g^{11} = 0.$$

This finishes the proof of Claim 1.

**Claim 2:**  $D_{ijk} = 0$  implies  $C_{1ab} = C_{a1b} = 0$ .

To do so, let us compute the second fundamental form in the preferred local coordinates system  $(f, \theta^2, \dots, \theta^n)$ :

$$h_{ab} = -\langle \nu, \nabla_a \partial_b \rangle = -\langle \nu, \Gamma_{ab}^1 \partial_f \rangle = \frac{\Gamma_{ab}^1}{|\nabla f|}.$$

But the Christoffel symbol  $\Gamma_{ab}^1$  is given by

$$\Gamma_{ab}^1 = \frac{1}{2}g^{11}\left(-\frac{\partial g_{ab}}{\partial f}\right) = \frac{1}{2}|\nabla f|\nu(g_{ab}).$$

Hence, we obtain

$$h_{ab} = \frac{1}{2}\nu(g_{ab}). \quad (4.6)$$

On the other hand, since  $|\nabla f|$  is constant along level surfaces, we have

$$[\partial_a, \nu] = -[\partial_a, |\nabla f|\partial_f] = 0.$$

Then using the fact that  $\langle \nu, \nu \rangle = 1$  and  $\langle \nu, \partial_a \rangle = 0$ , it is easy to see that

$$\nabla_\nu \nu = 0. \quad (4.7)$$

By direct computations and using Proposition 3.2, we can compute the following component of the Riemannian curvature tensor:

$$\begin{aligned}
Rm(\nu, \partial_a, \nu, \partial_b) &= \langle \nabla_\nu \nabla_a \partial_b - \nabla_a \nabla_\nu \partial_b, \nu \rangle \\
&= \langle \nabla_\nu (\nabla_a^\Sigma \partial_b + \nabla_a^\perp \partial_b), \nu \rangle - \langle \nabla_a \nabla_\nu \partial_b, \nu \rangle \\
&= \langle \nabla_a^\Sigma \partial_b, -\nabla_\nu \nu \rangle + \langle \nabla_\nu (-h_{ab} \nu), \nu \rangle + \langle \nabla_b \nu, \nabla_a \nu \rangle \\
&= -\nu(h_{ab}) + h_{ac} h_{cb} \\
&= -\frac{\nu(H)}{n-1} g_{ab} + \frac{H^2}{(n-1)^2} g_{ab}.
\end{aligned}$$

Taking trace in  $a, b$  yields

$$Rc(\nu, \nu) = -\nu(H) + \frac{H^2}{n-1}.$$

Thus

$$\begin{aligned}
Rm(\nu, \partial_a, \nu, \partial_b) &= -\frac{\nu(H)}{n-1} g_{ab} + \frac{H^2}{(n-1)^2} g_{ab} \\
&= \frac{Rc(\nu, \nu)}{n-1} g_{ab}.
\end{aligned}$$

Finally, we are ready to compute  $C_{1ab}$ :

$$C_{1ab} = -W_{1abi} \nabla_j f g^{ij} = W_{1a1b} |\nabla f|^2 = W(\nu, \partial_a, \nu, \partial_b). \quad (4.8)$$

However, by using proposition 3.2(e), we have:

$$\begin{aligned}
W(\nu, \partial_a, \nu, \partial_b) &= Rm(\nu, \partial_a, \nu, \partial_b) + \frac{Rg_{ab}}{(n-1)(n-2)} - \frac{1}{n-2} (Rc(\nu, \nu) g_{ab} + R_{ab}) \\
&= \frac{Rc(\nu, \nu)}{n-1} g_{ab} + \frac{Rg_{ab}}{(n-1)(n-2)} - \frac{1}{n-2} (Rc(\nu, \nu) g_{ab} + R_{ab}) \\
&= \frac{\lambda}{n-1} g_{ab} + \frac{(\lambda + (n-1)\mu) g_{ab}}{(n-1)(n-2)} - \frac{1}{n-2} (\lambda g_{ab} + \mu g_{ab}) \\
&= 0.
\end{aligned}$$

Hence,

$$C_{1ab} = W_{1a1b} = 0. \quad (4.9)$$

This finishes the proof of Claim 2.

Therefore we have shown that  $C_{ij1} = 0$ ,  $C_{abc} = 0$  and  $C_{1ab} = 0$ . This proves Lemma 4.2.  $\square$

For dimension  $n = 4$ , we can prove a stronger result:

**Lemma 4.3.** *Let  $(M^4, g_{ij}, f)$  be a complete gradient shrinking Ricci soliton with vanishing  $D_{ijk}$ , then the Weyl tensor  $W_{ijkl} = 0$  at all points where  $\nabla f \neq 0$ .*

*Proof.* From Lemma 4.2 we know that  $D_{ijk} = 0$  implies  $C_{ijk} = 0$ . Hence it follows from Lemma 3.1 that

$$W_{ijkl} \nabla_l f = 0$$

for all  $1 \leq i, j, k, l \leq 4$ . For any  $p$  where  $|\nabla f| \neq 0$ , we can attach an orthonormal frame at  $p$  with  $e_1 = \frac{\nabla f}{|\nabla f|}$ , and then we have

$$W_{1ijk}(p) = 0, \quad \text{for } 1 \leq i, j, k \leq 4. \quad (4.10)$$

Thus it remains to show

$$W_{abcd}(p) = 0$$

for all  $2 \leq a, b, c, d \leq 4$ . However, this essentially reduces to showing the Weyl tensor is zero in 3 dimensions (cf. [17], p.276–277): observing that the Weyl tensor  $W_{ijkl}$  has all the symmetry of the  $R_{ijkl}$  and is trace free in any two indices. Thus,

$$W_{2121} + W_{2222} + W_{2323} + W_{2424} = 0,$$

and so, by (4.10),

$$W_{2323} = -W_{2424}.$$

Similarly, we have

$$W_{2424} = -W_{3434} = W_{2323},$$

which implies  $W_{2323} = 0$ . On the other hand,

$$W_{1314} + W_{2324} + W_{3334} + W_{4344} = 0,$$

so  $W_{2324} = 0$ . This shows that  $W_{abcd} = 0$  unless  $a, b, c, d$  are all distinct. But, there are only three choices for the indices  $a, b, c, d$  as they range from 2 to 4.  $\square$

Now we are ready to finish the proof of our main theorems:

**Conclusion of the proof of Theorem 1.1:** Let  $(M^4, g_{ij}, f)$  be a complete Bach-flat gradient shrinking Ricci soliton. Then, by Lemma 4.1, we know  $D_{ijk} = 0$ . We divide the arguments into two cases:

- Case 1: the set  $\Omega = \{p \in M | \nabla f(p) \neq 0\}$  is dense.

By Lemma 4.1 and Lemma 4.3, we know that  $W_{ijkl} = 0$  on  $\Omega$ . By continuity, we know that  $W_{ijkl} = 0$  on  $M^4$ . Therefore we conclude that  $(M^4, g_{ij}, f)$  is locally conformally flat. Furthermore, according to the classification result for locally conformally flat gradient shrinking Ricci solitons mentioned in the introduction,  $(M^4, g_{ij}, f)$  is a finite quotient of either  $\mathbb{R}^4$ , or  $\mathbb{S}^3 \times \mathbb{R}$ .

- Case 2:  $|\nabla f|^2 = 0$  on some nonempty open set. In this case, since any gradient shrinking Ricci soliton is analytic in harmonic coordinates, it follows that  $|\nabla f|^2 = 0$  on  $M$ , i.e.,  $(M^4, g_{ij})$  is Einstein.

This completes the proof of Theorem 1.1.  $\square$

**Conclusion of the proof of Theorem 1.2:** Let  $(M^n, g_{ij}, f)$ ,  $n \geq 5$ , be a Bach-flat gradient shrinking Ricci soliton. Then, by Lemma 4.1, Lemma 4.2 and the same argument as in the proof of Theorem 1.1 above, we know that  $(M^n, g_{ij}, f)$  either is Einstein, or has harmonic Weyl tensor. In the latter case, by the rigidity theorem of Fernández-López and García-Río [16] and Munteanu-Sesum [20] for harmonic Weyl tensor,  $(M^n, g_{ij}, f)$  is either Einstein or isometric to a finite quotient of

$N^{n-k} \times \mathbb{R}^k$  ( $k > 0$ ) the product of an Einstein manifold  $N^{n-k}$  with the Gaussian shrinking soliton  $\mathbb{R}^k$ . However, Proposition 3.2 (e) says that the Ricci tensor either has one unique eigenvalue or two distinct eigenvalues with multiplicity of 1 and  $n - 1$  respectively. Therefore, only  $k = 1$  and  $k = n$  can occur in  $N^{n-k} \times \mathbb{R}^k$ .  $\square$

## 5. GRADIENT RICCI SOLITONS WITH VANISHING $D_{ijk}$

First of all, we notice that the proofs of Lemma 4.2 and Lemma 4.3 are valid for gradient steady and expanding Ricci solitons as well. Hence we have the following general result.

**Theorem 5.1.** *Let  $(M^n, g_{ij}, f)$  ( $n \geq 4$ ) be a complete non-trivial gradient Ricci soliton satisfying (3.1) and with  $D_{ijk} = 0$ , then*

- (i) *the Weyl tensor  $W_{ijkl} = 0$  for  $n = 4$ , i.e.,  $(M^n, g_{ij}, f)$  is locally conformally flat;*
- (ii) *the Cotton tensor  $C_{ijk} = 0$  for  $n \geq 5$ , i.e.,  $(M^n, g_{ij}, f)$  has harmonic Weyl tensor.*

As an immediate consequence of Theorem 5.1, the classification theorem for locally conformally flat gradient shrinking solitons and the rigidity theorem for gradient shrinking solitons with harmonic Weyl tensor mentioned in the introduction, and Proposition 3.2 (e), we have the following rigidity theorem for gradient shrinking Ricci solitons with vanishing  $D_{ijk}$ :

**Corollary 5.1.** *Let  $(M^n, g_{ij}, f)$  ( $n \geq 4$ ) be a complete gradient shrinking Ricci soliton with  $D_{ijk} = 0$ , then*

- (i)  *$(M^4, g_{ij}, f)$  is either Einstein, or a finite quotient of  $\mathbb{R}^4$  or  $\mathbb{S}^3 \times \mathbb{R}$ ;*
- (ii) *for  $n \geq 5$ ,  $(M^n, g_{ij}, f)$  is either Einstein, or a finite quotient of the Gaussian shrinking soliton  $\mathbb{R}^n$ , or a finite quotient of  $N^{n-1} \times \mathbb{R}$ , where  $N^{n-1}$  is Einstein.*

Moreover, combining Theorem 5.1 (i) and the 4-d classification theorem for locally conformally flat gradient steady Ricci solitons [8, 11], we have

**Corollary 5.2.** *Let  $(M^4, g_{ij}, f)$  be a complete gradient steady Ricci soliton with  $D_{ijk} = 0$ , then  $(M^4, g_{ij}, f)$  is either Ricci flat or isometric to the Bryant soliton.*

Finally, let us further examine the relations among  $D_{ijk}$ ,  $C_{ijk}$ ,  $W_{ijkl}$  and  $B_{ij}$ . Note that Theorem 5.1(ii) tells us that for any nontrivial gradient Ricci soliton,  $D_{ijk} = 0$  implies  $C_{ijk} = 0$ . On the other hand, the converse is not true because the product space  $\mathbb{S}^k \times \mathbb{R}^{n-k}$  has  $C_{ijk} = 0$  but not  $D_{ijk} = 0$  by Proposition 3.2(e) for  $k \geq 2$  and  $n - k \geq 2$ . So one naturally would wonder how much stronger is the condition  $D_{ijk} = 0$  than  $C_{ijk} = 0$ ? It turns out that we have several equivalent characterizations of  $D_{ijk} = 0$ .

**Theorem 5.2.** *Let  $(M^n, g_{ij}, f)$  ( $n \geq 5$ ) be a nontrivial gradient Ricci soliton satisfying (3.1). Then the following statements are equivalent:*

- (a)  $D_{ijk} = 0$ ;
- (b)  $C_{ijk} = 0$ , and  $W_{1ijk} = 0$  for  $1 \leq i, j, k \leq n$ ;
- (c)  $\operatorname{div} B \cdot \nabla f = 0$  and  $W_{1a1b} = 0$  for  $2 \leq a, b \leq n$ .

*Proof.* (a)  $\rightarrow$  (b): This follows from Theorem 5.1 and Lemma 3.1.

(b)  $\rightarrow$  (c): Clearly, it suffices to show that  $C_{ijk} = 0$  implies  $\operatorname{div} B \cdot \nabla f = 0$ . In fact,  $C_{ijk} = 0$  implies  $\operatorname{div} B = 0$  for  $n \geq 5$ . This follows from the following formula, which is well-known at least for  $n = 4$  among experts in conformal geometry and general relativity:

**Lemma 5.1.** *For  $n \geq 4$ , we have*

$$\operatorname{div} B \equiv \nabla_j B_{ij} = \frac{n-4}{(n-2)^2} C_{ijk} R_{jk}.$$

*Proof of Lemma 5.1.* Recall that we have

$$C_{ijk} = \nabla_i A_{jk} - \nabla_j A_{ik},$$

and

$$W_{ijkl} = R_{ijkl} - \frac{1}{n-2} (g_{ik} A_{jl} - g_{il} A_{jk} - g_{jk} A_{il} + g_{jl} A_{ik}). \quad (5.1)$$

By using the expression of the Bach tensor in (2.3), we have

$$(n-2) \nabla_i B_{ij} = \nabla_i \nabla_k (\nabla_k A_{ij} - \nabla_i A_{kj}) + \nabla_k R_{kl} W_{ikjl} + R_{kl} \nabla_k W_{ikjl}.$$

But,

$$\begin{aligned} \nabla_i \nabla_k (\nabla_k A_{ij} - \nabla_i A_{kj}) &= (\nabla_i \nabla_k - \nabla_k \nabla_i) \nabla_k A_{ij} \\ &= -R_{il} \nabla_l A_{ij} + R_{kl} \nabla_k A_{lj} + R_{ikjl} \nabla_k A_{il} \\ &= R_{ikjl} \nabla_k A_{il}. \end{aligned}$$

Thus, by using (5.1),

$$\begin{aligned} \nabla_i \nabla_k (\nabla_k A_{ij} - \nabla_i A_{kj}) + \nabla_k R_{kl} W_{ikjl} &= (R_{ikjl} - W_{ikjl}) \nabla_k A_{il} \\ &= \frac{1}{n-2} (A_{jk} g_{il} C_{lki} + A_{ik} C_{kji}) \\ &= -\frac{1}{n-2} R_{ki} C_{jki}. \end{aligned}$$

Moreover, by (2.2), we know

$$\nabla_k W_{ikjl} = \frac{n-3}{n-2} C_{jlk}.$$

Summing up, we obtain

$$(n-2) \nabla_i B_{ij} = \frac{n-4}{n-2} R_{kl} C_{jkl}.$$

□

(c)  $\rightarrow$  (a): by Lemma 5.1, Lemma 3.1 and (3.3), we have

$$\begin{aligned} \operatorname{div} B \cdot \nabla f &= \frac{n-4}{(n-2)^2} C_{ijk} R_{jk} \nabla_i f \\ &= \frac{n-4}{(n-2)^2} (D_{ijk} - W_{ijkl} \nabla_l f) R_{jk} \nabla_i f \\ &= \frac{n-4}{2(n-2)} |D_{ijk}|^2 + \frac{n-4}{(n-2)^2} W_{1a1b} R_{ab} |\nabla f|^2. \end{aligned}$$

Thus,  $\operatorname{div} B \cdot \nabla f = 0$  and  $W_{1a1a} = 0$  for  $2 \leq a \leq n$  imply  $D_{ijk} = 0$  for all  $1 \leq i, j, k \leq n$ .

This completes the proof of Theorem 5.2.

□

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